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CERTAIN DYNAMIC AND STATIC CONTACT PROBLEMS OF THE THEORY OF ELASTICITY FOR A CIRCULAR CYLINDER OF FINITE SIZE*

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Axisymmetric, dynamic contact problems of the theory of elasticity concerning the vertical (problem 1) and torsional (problem 2) oscillations of a stamp lying on a plane boundary of a circular cylinder of finite size, are considered. In case of problem 1 it is assumed that the side surface of the cylinder is in contact with a smooth, rigid yoke, and for problem 2 the side surface of the cylinder is immovable. A static problem (problem 3) formulated analogously to problem 1 is also studied. The solutions of the above problems are obtained using the method of homogeneous solutions /1/. Conditions of generalized orthogonality of the axisymmetric homogeneous solutions are obtained for the problem of steady-state oscillations in a layer, in a manner analogous to that used in /2/. A numerical example is solved, which shows that in a static problem with the cylinder height and radius of the stamp both fixed, the resistance of the cylinder against the penetration of the stamp is a nonmonotonous function of the cylinder radius. Problem 1 was solved by a different method in /3/, and a number of axisymmetric contact problems for a cylinder formulated in a similar manner were dealt with in /4-9/ et al.

1. Condition of generalized orthogonality in the problem of steady-state oscillations of a layer. Let us consider an elastic layer $|z| \leqslant h$, $r \geqslant 0$ (r, z, φ are cylindrical coordinates), and let the edges $z = \pm h$ of this layer be a) fixed, b) stress-free, or the edge $z = h$ be fixed and $z = -h$ stress-free. Seeking a solution of the Lamé equations in the form

$$u_k(r, z) = A_k(z) J_1(p_k r), \quad w_k(r, z) = B_k(z) J_0(p_k r) \tag{1.1}$$

where $u_k(r, z)e^{i\omega t}$ and $w_k(r, z)e^{i\omega t}$ are the projections of the displacement vector on the r-and z-axis respectively, ω is the oscillation frequency and t is time, we obtain the system of differential equations

$$A_k'' + (\theta_2^2 - \alpha p_k^2)A_k - (1-2\nu)^{-1} p_k B_k' = 0, \quad \alpha B_k'' + (\theta_2^2 - p_k^2) B_k + (1-2\nu)^{-1} p_k A_k' = 0, \quad \theta_2^2 = \frac{\rho\omega^2}{\mu}, \tag{1.2}$$

$$\alpha = 2\frac{1-\nu}{1-2\nu}$$

under the conditions that

$$\text{a)}\ A_k(\pm h) = B_k(\pm h) = 0, \ \text{b)}\ \sigma_{zk}^*(\pm h) = \tau_k^*(\pm h) = 0, \text{c)}\ A_k(h) = B_k(h) = \sigma_{zk}^*(-h) = \tau_k^*(-h) = 0 \tag{1.3}$$

where ρ is density, μ and ν are the elastic constants of the material and the components of the stress tensor without the temporary multiplier have the form

$$\sigma_{zk}(r, z) = \mu\sigma_{zk}^*(z) J_0(p_k r), \quad \tau_{rzk}(r, z) = \mu\tau_k^*(z) J_1(p_k r), \quad \sigma_{rk} = \mu[\sigma_{rk}^*(z) J_0(p_k r) - 2A_k(z) r^{-1} J_1(p_k r)] \tag{1.4}$$

$$\sigma_{zk}^*(z) = \beta p_k A_k(z) + \alpha B_k'(z), \quad \tau_k^*(z) = A_k'(z) - p_k B_k(z), \quad \sigma_{rk}^*(z) = \alpha p_k A_k(z) + \beta B_k'(z)$$

Let the problem (1.2), (1.3) have simple eigenvalues only, and $p_j^2 \neq p_n^2$. Then its eigenfunctions will satisfy the following relations of generalized orthogonality (λ is the elastic constant of the material):

$$U_{jn} = \int_{-h}^{h} [p_j p_n B_j B_n + \theta_2^2 A_j A_n - A_j' A_n'] dz = 0, \quad V_{jn} = \int_{-h}^{h} [p_j p_n A_j A_n + \theta_1^2 B_j B_n - B_j' B_n'] dz = 0 \tag{1.5}$$

$$W_{jn} = \int_{-h}^{h} [\sigma_{rj}^* A_n - B_j \tau_n^*] dz = 0, \quad \theta_1^2 = \frac{\rho\omega^2}{\lambda + 2\mu}$$

The first two relations of (1.5) are obtained in a manner analogous to that used in /2/ for a plane problem. It can also be shown that

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$$W_{jn} = 2(p_n^2 - p_j^2)^{-1}[p_n\sigma_{zn}^*(z)B_j(z) - p_nB_n(z)\sigma_{zj}^*(z) - p_j\tau_j^*(z)A_n(z) + p_jA_j(z)\tau_n^*(z)]|_{-h}^{h} \quad (1.6)$$

from which the last relation of (1.5) follows with any form of the boundary condition (1.3) taken into account.

2. Vertical oscillations of a stamp.

Let us consider an axisymmetric contact problem of vertical nonresonant oscillations of a stamp of radius a, lying without friction on a plane boundary of a circular cylinder of radius R and height h, acted upon by a vertical force $Pe^{-i\omega t}$, under the following boundary conditions:

$$\begin{gathered}\sigma_z(r,z)=0\ \ (z=h,\ a<r<R),\quad w(r,z)=\delta(r)\ \ (z=h,\ r\leqslant a)\\ \tau_{rz}(r,z)=0\ \ (z=h,\ z=0,\ r\leqslant R),\quad w(r,z)=0\ \ (z=0,\ r\leqslant R)\\ \tau_{rz}(r,z)=u(r,z)=0\ \ (r=R,\ 0\leqslant z\leqslant h)\end{gathered} \quad (2.1)$$

Here $ue^{-i\omega t}$, $we^{-i\omega t}$ are the projections of the displacement vector on the r-and z-axis respectively, and $\sigma_z e^{-i\omega t}$, $\tau_{rz}e^{-i\omega t}$ are the components of the stress tensor. We solve the problem stated using the method of homogeneous solutions /1/. According to this method we first find a solution of the problem for the layer when

$$\begin{gathered}\sigma_z(r,z)=q(r)\ \ (z=h,\ r\leqslant a),\ \ \sigma_z(r,z)=0\ \ (z=h,\ r>a)\\ \tau_{rz}(r,z)=0\ \ (z=h),\ \ \tau_{rz}(r,z)=w(r,z)=0,\ \ (z=0)\end{gathered} \quad (2.2)$$

Using the principle of limiting absorption /10/, we multiply the right-hand sides of the Lamé equations by the corresponding weighing terms

$$\varepsilon\rho\omega\partial(ue^{-i\omega t})/\partial t,\quad \varepsilon\rho\omega\partial(we^{-i\omega t})/\partial t$$

(ε is the fictitious absorption coefficient) and seek a solution of such equations in the form $ue^{-i\omega t}$, $we^{-i\omega t}$. Separating the variables and applying a Hankel transform to the resulting equations in u and w, we obtain

$$w^{(1)}(r,z)=\frac{1}{\mu}\int_0^a q(\rho)\rho\,d\rho\int_0^\infty L_\varepsilon(z,u)J_0(ur)J_0(u\rho)u\,du,\quad u^{(1)}(r,z)=\frac{1}{\mu}\int_0^a q(\rho)\rho\,d\rho\int_0^\infty L_{1\varepsilon}(z,u)J_1(ur)J_0(u\rho)u\,du \quad (2.3)$$

$$\tau_{rz}^{(1)}(r,z)=\int_0^a q(\rho)\rho\,d\rho\int_0^\infty L_{2\varepsilon}(z,u)J_1(ur)J_0(u\rho)u\,du$$

$$L_\varepsilon(z,u)=B(z,u)\gamma(u),\quad L_{1\varepsilon}(z,u)=A(z,u)\gamma(u)$$

$$L_{2\varepsilon}(z,u)=[A'(z,u)-uB(z,u)]\gamma(u),\quad \gamma(u)=[\beta uA(h,u)+\alpha B'(h,u)]^{-1}$$

$$A(z,u)=u\varkappa_\varepsilon^{-1}[(\eta_\varepsilon^2+u^2)\operatorname{sh}\eta_\varepsilon h\operatorname{ch}\varkappa_\varepsilon z-2\varkappa_\varepsilon\eta_\varepsilon\operatorname{ch}\eta_\varepsilon z\operatorname{sh}\varkappa_\varepsilon h]$$

$$B(z,u)=-[(\eta_\varepsilon^2+u^2)\operatorname{sh}\eta_\varepsilon h\operatorname{sh}\varkappa_\varepsilon z-2u^2\operatorname{sh}\eta_\varepsilon z\operatorname{sh}\varkappa_\varepsilon h]$$

$$\varkappa_\varepsilon^2=u^2-\rho\omega^2(1+i\varepsilon)/(\lambda+2\mu),\quad \eta_\varepsilon^2=u^2-\rho\omega^2(1+i\varepsilon)/\mu$$

where a prime denotes a derivative with respect to its first argument.

In the second stage of the solution we construct a system of homogeneous solutions of the Lamé equations, transformed in the manner shown above, for a layer. We then have

$$\sigma_z(r,z)=\tau_{rz}(r,z)=0\ \ (z=h),\quad w(r,z)=\tau_{rz}(r,z)=0,\ \ (z=0)$$

The above boundary conditions are equivalent to conditions b) given in Sect.1, provided that the boundary value problem is continued symmetrically into the region $-h\leqslant z<0$. The projections of the displacement vector and the components of the stress tensor will have the form (1.1), (1.4), where

$$A_k(z)=A(z,p_k),\quad B_k(z)=B(z,p_k) \quad (2.4)$$

and we must replace $\varkappa_\varepsilon^2$ and η_ε^2 by

$$\varkappa_{k\varepsilon}^2=p_k^2-\rho\omega^2(1+i\varepsilon)/(\lambda+2\mu),\quad \eta_{k\varepsilon}^2=p_k^2-\rho\omega^2(1+i\varepsilon)/\mu,\quad (\beta p_kA_k(h)+\alpha B_k'(h)=0)$$

respectively, where p_k are the roots of the equation contained within the brackets.

In the third stage we introduce the functions

$$u^{(2)}(r,z)=\sum_{k=1}^\infty D_kA_k(z)J_1(p_kr),\quad w^{(2)}(r,z)=\sum_{k=1}^\infty D_kB_k(z)J_0(p_kr),\quad \tau_{rz}^{(2)}(r,z)=\sum_{k=1}^\infty D_k\tau_k^*(z)J_1(p_kr) \quad (2.5)$$

where the summation is carried out over all p_k for which $\operatorname{Im}(p_k)>0$, and D_k are unknown coefficients. Then we can write the solution of the problem formulated at this stage in the form

$$u(r, z) = u^{(1)}(r, z) - u^{(2)}(r, z), \quad w(r, z) = w^{(1)}(r, z) - w^{(2)}(r, z) \tag{2.6}$$

We find the coefficients D_k of the expansion (2.5) from the condition

$$u(r, z) = 0, \ \tau_{rz}(r, z) = \tau_{rz}^{(1)}(r, z) - \tau_{rz}^{(2)}(r, z) = 0 \quad (r = R)$$

which we shall rewrite thus

$$\sum_{k=1}^{\infty} D_k A_k(z) J_1(p_k R) = u^{(1)}(R, z), \quad \sum_{k=1}^{\infty} D_k \tau_k^*(r) J_1(p_k R) = \mu^{-1} \tau_{rz}^{(1)}(R, z)$$

Let us multiply the first equation of the above relations by $\sigma_{rj}^*(z)$, the second by $B_j(z)$, subtract the second from the first and integrate from $-h$ to h. Taking into account the last relation of (1.5) which holds also when $\varepsilon \neq 0$, we obtain

$$D_k = (\mu W_{kk} J_1(p_k R))^{-1} \int_{-h}^{h} [\mu u^{(1)}(R, z) \sigma_{rk}^*(z) - \tau_{rz}^{(1)}(R, z) B_k(z)]\, dz = (\mu W_{kk} J_1(p_k R))^{-1} \int_0^a q(\rho) \Omega_k(\rho) \rho\, d\rho \tag{2.7}$$

$$\Omega_k(\rho) = \int_0^{\infty} M_k(u) J_1(uR) J_0(u\rho)\, u\, du, \quad M_k(u) = \gamma(u) \int_{-h}^{h} [A(z, u) \sigma_{rk}^*(z) - (A'(z, u) - uB(z, u)) B_k(z)]\, dz$$

Equating the last relations of (2.7) and (1.5) we obtain $M_k(p_j) = \gamma(p_j) W_{kj}$, where the function $\gamma(u)$ is given by (2.3). Taking into account the boundary conditions for the homogeneous solutions and the relation (1.6) for W_{kj}, we obtain

$$M_k(u) = 2uB_k(h)\ (u^2 - p_k^2)^{-1}, \quad \Omega_k(\rho) = 2B_k(h) \int_0^{\infty} \frac{u^2}{u^2 - p_k^2} J_1(uR) J_0(u\rho)\, du = -2 B_k(h)\, i p_k I_0(-i\rho p_k) K_1(-iRp_k)$$

The last integral is taken from /11/, taking into account the fact that $\operatorname{Im}(p_k) \neq 0$ for all p_k: $I_0(x)$ and $K_1(x)$ are modified Bessel functions.

We find now that all conditions (2.1) of the problem 1 hold, with exception of the condition

$$w(r, z) = w^{(1)}(r, z) - w^{(2)}(r, z) = \delta(r) \quad (z = h, \ r \leqslant a)$$

Let us introduce the operator $K_{rh}^{\varepsilon} q = \mu w^{(1)}(r, h)$, where $w^{(1)}(r, h)$ is defined by one of the formulas of (2.3). Then, satisfying the last condition, we obtain the following integral equation for the contact pressure $q(\rho)$ under the stamp:

$$\mu^{-1} K_{rh}^{\varepsilon} q = \delta(r) + \sum_{k=1}^{\infty} D_k B_k(h) J_0(p_k r) \quad (r \leqslant a)$$

Writing now $q(\rho)$ in the form

$$q(\rho) = \frac{\mu}{1 - \nu} \left[q_0(\rho) + \sum_{k=1}^{\infty} D_k B_k(h) q_k(\rho) \right] \tag{2.8}$$

where $q_k(\rho)$ is the solution of the integral equations

$$K_{rh}^{\varepsilon} q_0 = (1 - \nu)\delta(r) \quad (r \leqslant a), \quad K_{rh}^{\varepsilon} q_k = (1 - \nu) J_0(p_k r), \quad (k \geqslant 1, \ r \leqslant a) \tag{2.9}$$

and substituting (2.8) into (2.7), we obtain an infinite system of linear algebraic equations for determining the constants D_k of the expansion (2.8):

$$x_k = g_k + \sum_{n=1}^{\infty} a_{kn} x_n \ \ (x_k = D_k B_k(h) I_1(R\gamma_k / h), k \geqslant 1), \ \ a_{kn} = -2i\gamma_k W_{kk}^{-1} B_k^2(h) K_1(R\gamma_k / h) I_1^{-1}(R\gamma_n / h)\, T_{n,k} \tag{2.10}$$

$$g_k = -2i\gamma_k W_{kk}^{-1} B_k^2(h) K_1(R\gamma_k / h) T_{0,k}, \quad \gamma_k = -ip_k h, \quad T_{n,k} = \int_0^{\infty} q_n(\rho) I_0(\rho\gamma_k / h) \rho\, d\rho$$

Until now we assumed that the coefficient of fictitious absorption of the medium $\varepsilon > 0$. Making ε tend to zero, we obtain a solution of the initial problem 1. It must be remembered here /12/ that some of the zeros and poles of the function $L_\varepsilon(h, u)$ given in (2.3) will pass, as $\varepsilon \to 0$, to the real axis, and this will distort the contour of integration in the expression for the kernel of the integral equations (2.9). The authors of /12/ discuss the shape of such a contour Γ in detail.

If follows that the contact pressure is defined by the formula

$$q(\rho) = \frac{\mu}{1 - \nu} \left[q_0(\rho) + \sum_{k=1}^{\infty} x_k I_1^{-1}(\gamma_k R / h) q_k(\rho) \right] \tag{2.11}$$

where x_k is a solution of the system (2.1) for $\varepsilon = 0$, $q_k(\rho)$ is a solution of the known /12/ integral equations $(q_k(a\rho) = \vartheta_k(\rho))$ written in dimensionless variables

$$\int_0^1 \vartheta_k(\rho)\rho\, d\rho \int_\Gamma L(u\lambda) J_0(ur) J_0(u\rho)\, du = f_k(r) \quad (r \leqslant 1) \tag{2.12}$$

$$L(\tau) = \frac{\theta_2^{*2}(1-\nu)^{-1}\tau\varkappa\,\mathrm{sh}\,\varkappa\,\mathrm{ch}\,\eta}{4\tau^2\varkappa\eta\,\mathrm{ch}\,\eta\,\mathrm{sh}\,\varkappa - (\eta^2+\tau^2)^2\,\mathrm{sh}\,\eta\,\mathrm{ch}\,\varkappa}, \quad \lambda = \frac{h}{a}, \; \varkappa^2 = \tau^2 - \theta_1^{*2}, \; \eta^2 = \tau^2 - \theta_2^{*2}, \; \theta_1^{*2} = \frac{\rho\omega^2 h}{\lambda + 2\mu}, \; \theta_2^{*2} = \frac{\rho\omega^2 h}{\mu} \tag{2.13}$$

$$f_k(r) = \{\delta(ra), \quad \text{if} \quad k=0; \quad I_0(a\gamma_k r/h), \quad \text{if} \quad k \geqslant 1\} \tag{2.14}$$

Moreover we shall assume in (2.12) that γ_k are poles of the function $L(\tau)$ (2.13). The contour Γ coincides with the positive part of the real axis everywhere except on the segments containing real poles of the function $L(\tau)$ /12/. In the case of alternating the zeros and poles of this function, the segments indicated are bypassed by the contour from below /12/.

Let us investigate the infinite system (2.10). We know /12/ that the function $L(\tau)$ has a finite number of real zeros and poles and that the number increases with increasing reduced frequency θ_2^*. At large numerical values the complex poles of the function $L(\tau)$ have the following asymptotic representation (*) (a_i ($i = 1, 2, 3, 4$) are real constants)

$$z_n = ih\gamma_n \sim ina_1 + a_2 \ln(a_3 n + a_4) \tag{2.15}$$

Taking into account (2.15) we can show as was done in /1,13/, that at large numerical values the coefficients of the infinite system (2.10) have the following asymptotics ($k, n \to \infty$):

$$|g_k| \sim k^{-1}\exp[-a_1 k(R-a)/h], \quad |a_{kn}| \sim k^{-1}\exp[-a_1(k+n)(R-a)/h] \tag{2.16}$$

It follows therefore that the system (2.10) belongs to the class of the normal Poincaré–Koch systems and can be solved by the reduction method for any value of the parameter $(R-a)/h > 0$.

3. Torsional oscillations of a stamp. We shall consider an axisymmetric contact problem of nonresonant torsional oscillations of a stamp of radius a, rigidly coupled to the plane boundary of a circular cylinder of radius R and height h, acted upon by the moment $Me^{-i\omega t}$, with the following boundary conditions:

$$v(r, z) = \delta r \; (r \leqslant a, z = h), \quad \tau_{z\varphi}(r, z) = 0 \quad (a < r < R, z = h) \tag{3.1}$$
$$v(r, z) = 0 \quad (z = 0, \; r \leqslant R \; \text{ and } \; r = R, \; 0 \leqslant z \leqslant h)$$

Here $ve^{-i\omega t}$ denotes the displacement along the φ-axis, $\tau_{z\varphi}e^{-i\omega t}$ are the tangential stresses and δ is the stamp oscillation amplitude.

Using the method of homogeneous solutions which was used to solve an analogous static problem in /13/, we reduce the present problem to that of investigating the infinite system (2.10) with the coefficients

$$g_k = 2(-1)^k K_1(R\gamma_k/h)T_{0,k}, \quad a_{kn} = 2(-1)^{k+n} h^{-1} K_1(R\gamma_k/h) I_1^{-1}(R\gamma_1/h] T_{n,k} \tag{3.2}$$

$$T_{n,k} = \int_0^a \tau_n(\rho) I_1(\rho\gamma_k/h)\rho\, d\rho, \quad i\gamma_k = [\varkappa^2 - \pi^2(k - {}^1/_2)^2]^{1/2}, \quad L(u) - (\sqrt{u^2-\varkappa^2})^{-1} u \,\mathrm{th}\sqrt{u^2-\varkappa^2}, \quad \varkappa^2 = \rho\omega^2 h^2\mu^{-1}$$
$$f_k(x) = \{\delta x \quad \text{if} \quad k = 0; \; a^{-1} I_1(a\gamma_k x/h) \quad \text{if} \quad k \geqslant 1\}$$

Here $\tau_n(a\rho) = \vartheta_n(\rho)$ are solutions of the integral equations (2.12) for $n = 1$, $z_k = i\gamma_k$ are the poles of the function $L(u)$, and the contour was chosen according to Sect.2. The tangential contact stresses under the stamp are defined by the formula

$$\tau(r) = \mu\tau_0(r) + \frac{\mu}{h}\sum_{k=1}^{\infty} x_k(-1)^k \tau_k(r) I_1^{-1}(R\gamma_k/h) \tag{3.3}$$

The asymptotic expressions (2.16) where $a_1 = \pi$ hold also for the coefficients of (3.2), therefore the system (2.10) with the coefficients (3.2) belongs to the normal Poincaré–Koch systems.

4. Static contact problem. Consider an axisymmetric static contact problem of imbedding a stamp of radius a into a plane boundary of a circular cylinder of radius R and height h, using a force P. The boundary conditions have the form (2.1) where u and w are projections of the displacement vector, and σ_z, τ_{rz} are components of the stress tensor. Again, as in Problem 1, we use the method of homogeneous solutions to find the contact pressure according to the formula (2.11) in which x_k is the solution of a system of the form (2.10) with the coefficients given by

* Makhema V.K. Three-dimensional dynamic problems of steady-state oscillation of plates. Avtoref. Kand. dis. Rostov-on-Don, 1979.

$$g_k = \text{tg}^2\,\gamma_k K_1\,(R\gamma_k/h)T_{0,k}, \quad a_{kn} = h^{-1}\,\text{tg}^2\gamma_k K_1\,(R\gamma_k/h)I_1^{-1}\,(R\gamma_k/h)\;T_{n,k} \tag{4.1}$$

$$L(u) = (\text{ch}\,2u - 1)\,(2u + \text{sh}2u)^{-1} \tag{4.2}$$

The function $T_{n,k}$ is given in (2.10), $q_k\,(a\rho)$ represents a solution of the integral equation (2.12) in which $f_k\,(r)$ has the form (2.14) and $i\gamma_k$ are complex poles of the function $L\,(\tau)$ lying in the upper half-plane. Since the function $L\,(\tau)$ has no real poles, it follows that the contour Γ in (2.12) will fully coincide with the positive part of the real axis. The infinite system (2.10) – (4.1) will, in this case, also belong to the normal Poincaré – Koch systems.

5. Solution of the integral equations (2.12). Contact problems for an elastic layer analogous to the Problems 1 – 3, can be reduced to integral equations of the type (2.12). Such equations have been exhaustively studied, and their solutions can be obtained using e.g. asymptotic methods /14/. We know (see e.g. /15/) that, when $\lim L\,(\tau) = 1 + O\,(\tau^{-2})$ $(\tau \to 0)$ the equation (2.12) is equivalent to the integral equation of the second kind

$$\varphi_k\,(t) = \frac{1}{\pi\lambda}\int_{-1}^{1}\varphi_k\,(\tau)\,M\left(\frac{t-\tau}{\lambda}\right)d\tau + d_k\,(t) \quad (|\,t\,| \leqslant 1) \tag{5.1}$$

$$M\,(y) = \int_{\Gamma}[1 - L\,(u)]\cos uy\,du \tag{5.2}$$

where $L\,(u)$ is (2.13) (Problem 1), (3.3) (Problem 2) or (4.2) (Problem 3). In the case of dynamic problems the contour Γ is situated as in Sects.2 and 3, and for the static problem it coincides with the positive part of real axis. Moreover, for Problems 1 and 3 we have

$$\vartheta_k\,(r) = -\frac{2}{\pi}\frac{d}{dr}\,r\int_{r}^{1}\frac{\varphi_k\,(\tau)\,d\tau}{\tau\sqrt{\tau^2 - r^2}}, \quad d_k\,(t) = \frac{d}{dt}\int_{0}^{t}\frac{rf_k\,(r)\,dr}{\sqrt{t^2 - r^2}} \tag{5.3}$$

and for Problem 2 we have

$$\vartheta_k\,(r) = -\frac{2}{\pi}\frac{d}{dr}\int_{r}^{1}\frac{\varphi_k\,(t)\,dt}{\sqrt{t^2 - r^2}}, \quad d_k\,(t) = \frac{d}{dt}\,t\int_{0}^{t}\frac{f_k\,(r)\,dr}{\sqrt{t^2 - r^2}} \tag{5.4}$$

We use the method of large λ (see e.g. /16/) to solve the integral equation (5.1), (5.2). To do this, we must write the kernel (5.2) in the form of an expansion in positive powers of $|\,y\,|$. This is easily done for Problem 3 /14/

$$M\,(y) = \sum_{k=0}^{\infty} b_k y^{2k}, \quad b_k = \frac{(-1)^k}{(2k)!}\int_{0}^{\infty}[1 - L\,(u)]\,u^{2k}\,du \tag{5.5}$$

and in this case we can write the solution of the equation (5.1), (5.5) for large λ in the form /13,16/

$$\varphi_m\,(t) = d_m\,(t) + \sum_{j=0}^{M} H_j{}^m t^{2j}, \quad H_j{}^m\,(\lambda) = \sum_{s=j}^{M}\lambda^{-(2s+1)}[\beta_{sj}^m + \lambda^{-1}d_{s+1,j}^m] \tag{5.6}$$

where M is an arbitrarily large number and the coefficients $\beta_{sj}{}^m$ and $\alpha_{sj}{}^m$ are found from the simple recurrence relations

$$\alpha_{sj}{}^m = \frac{2}{\pi}\sum_{k=j}^{s-1} b_k z_{kj}\sum_{p=0}^{s-k-1}\frac{\beta_{s-k-1,\,p}^m}{2p + 2k - 2j + 1} \quad \begin{pmatrix} s \geqslant 1 \\ 0 \leqslant j \leqslant s-1 \end{pmatrix} \tag{5.7}$$

$$\beta_{sj}{}^m = \frac{2}{\pi}\left[a^{-1}b_s z_{sj}F_{s-j}^m + \sum_{k=j}^{s-1} b_k z_{kj} \times \sum_{p=0}^{s-k-1}\frac{a_{s-k,\,p}^m}{2p + 2k - 2j + 1}\right] \quad \begin{pmatrix} s \geqslant 0 \\ 0 \leqslant j \leqslant s \end{pmatrix}$$

$$\beta_{00}{}^m = \frac{2}{\pi}\,b_0 z_{00} a^{-1} F_0{}^m, \quad z_{kj} = (2k)!\,[(2j)!\,(2k - 2j)!]^{-1}$$

$$F_k{}^0 = \delta\,(2k + 1)^{-1}, \quad F_k{}^m = \frac{(2k)!}{(a\gamma_m)^{2k+1}}\left[\sum_{i=0}^{k}\frac{(a\gamma_m)^{2i}}{(2i)!}\,\text{sh}\,a\gamma_m - \sum_{i=1}^{k}\frac{(a\gamma_m)^{2i-1}}{(2i-1)!}\,\text{ch}\,a\gamma_m\right] \quad (m \geqslant 1) \tag{5.8}$$

Thus in the case of large values of the parameter $\lambda\,(\lambda_0 < \lambda < \infty)$ we can solve (5.6) with any degree of accuracy.

For the kernel (5.2) of Problems 1 and 2 the following expansion holds:

$$M\,(y) = \sum_{k=0}^{\infty} b_k\,|\,y\,|^k \quad (0 \leqslant y \leqslant y_0 < \infty) \tag{5.9}$$

We shall show how this expansion can be obtained for Problem 2, since for Problem 1 the procedure will be exactly the same. Let us write $L(u)$ in the form $L(u) = L_1(u) + L_2(u)$ under the condition

$$L_2(u) = o(e^{-2u}), \quad L_1(u) = 1 - \sum_{i=1}^{\infty} c_i u^{-2i} \quad (u \to \infty) \tag{5.10}$$

This can be done if

$$L_1(u) = \frac{u}{\sqrt{u^2-\varkappa^2}}, \quad L_2(u) = \frac{u(\operatorname{th}\sqrt{u^2-\varkappa^2}-1)}{\sqrt{u^2-\varkappa^2}} \tag{5.11}$$

Then we have

$$M(y) = M_1(y) + M_2(y), \quad M_2(y) = \int_\Gamma L_2(u)\cos uy\,du = \sum_{k=0}^{\infty} y^{2k}(-1)^k[(2k)!]^{-1}\int_\Gamma L_2(u)u^{2k}\,du$$

$$M_1(y) = \int_\Gamma [1 - L_1(u)]\cos uy\,du = \sum_{k=0}^{\infty} b_k^* |y|^k, \quad b_{2k+1}^* = \frac{\pi(-1)^k c_{k+1}}{2(2k+1)!}, \quad b_{2k}^* = \frac{(-1)^k}{(2k)!}\int_\Gamma \left[1 - L(u) - \sum_{i=1}^{k}\frac{c_i}{u^{2i}}\right]u^{2k}\,du$$

$M_1(y)$ can be expanded into a series in the same manner as (1.3) in /17/. Thus the kernel (5.2) of the integral equation (5.1) can be written for Problems 1 and 2 in the form of a series (5.9), and solved using the method of large λ, with any degree of accuracy, in the form (5.6) where

$$H_j^m(\lambda) = \sum_{s=j}^{M} \lambda^{-2s}[\eta_{2s,j}^m + \lambda^{-1}\eta_{2s+1,j}^m] \tag{5.12}$$

and the coefficients η_{sj}^m are given by recurrent relations of the type (1.6) of /16/.

Knowing the solution of (5.1) in the form (5.6), and using the expressions (5.3) and (5.4), we can now obtain simple expressions for calculating the coefficients of the system (2.10) for Problems 1 and 2.

6. **Example.** We shall consider a static problem of imbedding a flat stamp $(\delta(r) = \delta = \text{const})$ into an elastic cylinder (Problem 3, Sect.4). We have the following contact stresses for this problem:

$$q(r) = \frac{\mu\delta}{1-\nu}\left[q_0(r) + \frac{1}{h}\sum_{k=1}^{\infty} x_k I_1^{-1}(\gamma_k R/h)\, q_k(r)\right] \quad (r \leqslant a), \quad q_k(\rho a) = \frac{2}{\pi}\left[\sum_{j=0}^{M} H_j^k(\lambda) S_j(\rho) + G_k(\rho)\right] \quad (\rho \leqslant 1,\ M \to \infty) \tag{6.1}$$

$$G_0(\rho) = a^{-1}S_0(\rho), \quad G_k(\rho) = a^{-1}\frac{d}{d\rho}\rho\int_\rho^1 \operatorname{ch}(a\gamma_m t/h)\, t^{-1}(t^2-\rho^2)^{-1/2}\,dt \quad (k \geqslant 1)$$

$$S_j(\rho) = \frac{1}{\sqrt{1-\rho^2}}\sum_{k=0}^{j-1}\frac{(j-1)!\,(2j\rho^2 - 2j + 2k + 1)}{k!\,(j-k-1)!\,(2k+1)}(1-\rho^2)^k \rho^{2(j-k-1)}$$

The relation connecting the force P acting on the stamp with the displacement δ of the stamp is given by the formulas

$$P = \frac{4a^2\mu\delta}{1-\nu}\left[P_0 + \frac{1}{h}\sum_{k=1}^{\infty} x_k I_1^{-1}(\gamma_k R/h) P_k\right], \quad P_k = R_k + \sum_{j=0}^{M} H_j^k(\lambda)(2j+1)^{-1}, \quad R_0 = \delta a^{-1}, R_k = ha^{-2}\gamma_k^{-1}\operatorname{sh}(a\gamma_k/h) \quad (k \geqslant 1) \tag{6.2}$$

where x_k is the solution of the infinite system (2.10) with coefficients (4.1), where

$$T_{n,k} = \frac{2a^2}{\pi}\left[\sum_{j=0}^{M} H_j^n(\lambda) F_j^k + t_{n,k}\right] \quad (M \to \infty), \quad t_{0,k} = a^{-2}h\gamma_k^{-1}\operatorname{sh}\frac{a\gamma_k}{h}, \quad t_{n,k} = \frac{h\operatorname{sh}[a(\gamma_n+\gamma_k)/h]}{2a^2(\gamma_n+\gamma_k)} + \frac{h\operatorname{sh}[a(\gamma_n-\gamma_k)/h]}{2a^2(\gamma_n-\gamma_k)} \tag{6.3}$$

The quantities F_j^k are given by the formulas (5.8), $H_j^k(\lambda)$ by (5.6) and (5.7), and γ_m denote the zeros of the function $2u + \sin 2u$ lying in the right half-plane $(\gamma_m \neq 0)$. Their asymptotic behavior at large values of m is known and given by /14/

$$\gamma_m \sim \pi(m - 1/4) \pm i/2\ln(4\pi m - \pi) \quad (m \to \infty)$$

A Fortran program was written for numerical solution using the computer BESM-6. The dimensionless quantities

$$P^* = P(1-\nu)(\mu\delta a)^{-1}, \quad q^*(\rho) = q(\rho a)(1-\nu)a(\mu\delta)^{-1} \quad (\rho \leqslant 1) \tag{6.4}$$

were studied for various values of the parameter λ and $R^* = R/a$. The values of P and $q(r)$ were found from (6.2) and (6.1), respectively. The proposed algorithm yields the values of

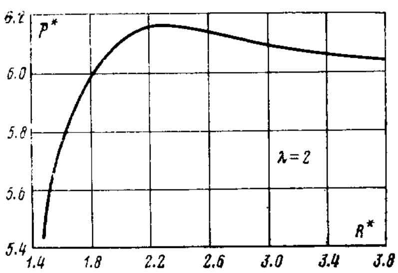

Fig.1

P^* and $q^*(\rho)$ for $\lambda \geqslant 1$ with practically any degree of accuracy, and the solution of the infinite system of linear algebraic equations is obtained using the reduction method. The quantity M of (5.6) was assumed finite in (6.1)—(6.3), and this enabled us to obtain P^* and $q^*(\rho)$ with an accuracy of up to the terms of the order of λ^{-2M-1}. We note that the larger the parameter $(R-a)/h$, the fewer equations of the reduced infinite system are required (see /13/ for a more detailed discussion of convergence of the proposed algorithm). Below we give the values of the quantities P^* and $q^*(\rho)$ for various λ, R^* and ρ:

$\lambda = 2$

R^*	1.5	2.0	2.2	2.4	2.6	2.8	3.0	∞
P^*	5.426	6.096	6.142	6.143	6.126	6.103	6.082	6.024
$q^*(0.20)$	0.971	1.022	1.029	1.032	1.034	1.035	1.036	1.039
$q^*(0.95)$	2.509	2.995	3.053	3.053	3.051	3.043	3.033	3.002

$\lambda = 4$

R^*	1.5	2.0	2.5	3.0	3.8	∞
P^*	2.948	4.075	4.616	4.844	4.940	4.882
$q^*(0.20)$	0.460	0.612	0.702	0.750	—	0.801
$q^*(0.95)$	1.351	2.001	2.311	2.444	—	2.480

Analysing the numerical values of P^* for fixed λ we can conclude that, when the parameter R^* increases from zero to some value depending on λ, the resistance of the cylinder against the imbedding of the stamp also increases. When the value of R^* is increased further, the resistance diminishes and tends to some constant value. This is illustrated in the Fig.1 where P^* is plotted against R^* for $\lambda = 2$. We note that the proposed algorithm yields solutions of the dynamic Problems 1 and 2 with any degree of accuracy also when $\lambda > \lambda^*(\omega)$, and in this case $\lambda^*(\omega)$ increases with the increasing frequency ω.

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